—Chapter 3—

Laplace's Equation

3-1 Laplace's Equation

A. LAPLACE'S EQUATION

(1) For an electrostatic field, we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \vec{E} = 0$$

and $\vec{E}$ goes to zero at infinity. If $\rho(r)$ is given, the electric field $\vec{E}$ is uniquely determined by

$$\vec{E} = -\nabla\varphi$$

where

$$\varphi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{|\vec{r}-\vec{r}'|} d^3r'$$

(2) From the differential form of Gauss's law, we obtain

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla\varphi) = -\nabla^2\varphi = \frac{\rho}{\epsilon_0}$$

$$\nabla^2\varphi = -\frac{\rho}{\epsilon_0} \cdots\cdots \text{Poisson equation}$$

where ∇^2 is called the Laplacian.

(3) In regions where there is no charge, so $\rho = 0$, we have

$$\nabla^2\varphi = 0 \cdots\cdots \text{Laplace's equation}$$

EXAMPLES:

1. The potential of a sphere

$$\varphi(r) = \begin{cases} \dfrac{\rho R^2}{2\epsilon_0} - \dfrac{\rho r^2}{6\epsilon_0}, & \text{inside the sphere} \\ \dfrac{\rho R^3}{3\epsilon_0 r}, & \text{outside the sphere} \end{cases}$$

Show the potential satisfying Laplace's equation.

ANSWER:

The Laplacian in spherical coordinates [c.f.3-3]:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Inside the sphere:

$$\nabla^2\varphi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\left(-\frac{\rho r^2}{6\epsilon_0}\right) = \frac{\rho}{6\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cdot -2r) = -\frac{\rho}{\epsilon_0}$$

satisfying Poisson's equation.

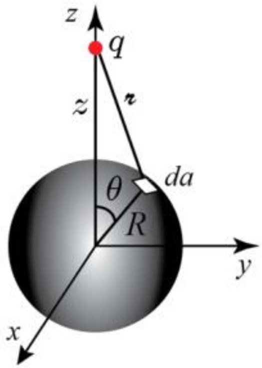
Outside the sphere:

$$\nabla^2\varphi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\left(\frac{\rho R^3}{3\epsilon_0 r}\right) = \frac{\rho R^3}{3\epsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot -\frac{1}{r^2}\right) = 0$$

satisfying Laplace's equation.

2. If φ satisfies Laplace's equation, then the average value of φ over the surface of any sphere is equal to the value of φ at the center of the sphere.

PROOF:



$$\begin{aligned}
\varphi_{\text{avg}} &= \frac{1}{4\pi R^2}\int \varphi\, da \\
&= \frac{1}{4\pi R^2}\int \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}R^2\sin\theta\, d\theta d\phi \\
&= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int_0^\pi \frac{R^2}{\sqrt{z^2+R^2-2zR\cos\theta}}\sin\theta\, d\theta\cdot 2\pi \\
&= \frac{q}{8\pi\epsilon_0 R^2}\int_0^\pi R\frac{d}{d\theta}\left(\frac{1}{z}\sqrt{z^2+R^2-2zR\cos\theta}\right)d\theta \\
&= \frac{q}{8\pi\epsilon_0 R}\frac{1}{z}[(z+R)-(z-R)] \\
&= \frac{q}{4\pi\epsilon_0 z}
\end{aligned}$$

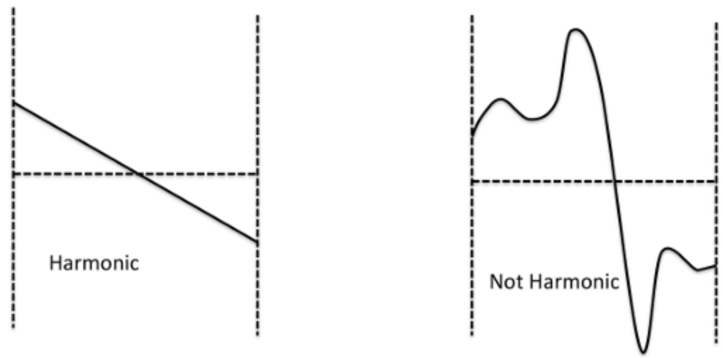
■

(4) A solution of Laplace's equation exists requiring a region (finite or

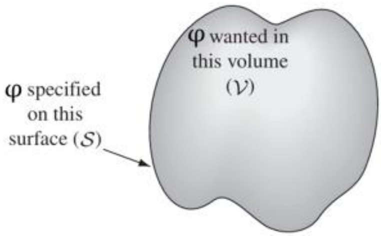
infinite), over which the differential equation is valid. This region has a boundary (could be infinite) on which a boundary condition is applied.

NOTE:

1. The second derivatives must exist throughout the region. This condition implies in turn that the first derivatives of φ are continuous. Any function that satisfies these conditions (and is thus a solution to Laplaces equation) is a HARMONIC function.
2. The second derivative must be greater than or less than zero for maxima or minima, but it is equal to zero for Laplace's equation. Harmonic functions have their maxima and minima at the boundaries of the region—any solution to Laplace's equation has no local minima or maxima, so the extrema must occur at the boudaries.



(5) The solution to Laplace's equation in some volume V is uniquely determined if φ is specified on the boundary surface S.



PROOF:

Suppose there are two solutions to Laplace's equation:

$$\nabla^2\varphi_1 = 0 \text{ and } \nabla^2\varphi_2 = 0$$

both of which assume the specified value on the same surface (boundary).

Let $\varphi_3 = \varphi_1 - \varphi_2$. We have $\nabla^2\varphi_3 = \nabla^2\varphi_1 - \nabla^2\varphi_2 = 0$

Since at the boundary, $\varphi_1 = \varphi_2$, so $\varphi_3 = \varphi_1 - \varphi_2 = 0$ at the boundary. However, Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. Therefore φ_3 must be zero everywhere, and hence

$$\varphi_1 = \varphi_2$$

When suitable boundary conditions are satisfied, the solutions to Laplace's equation are unique.

■

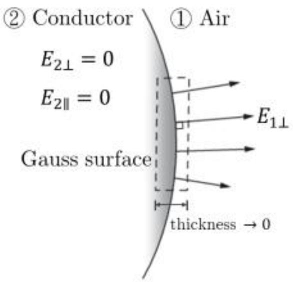
B. BOUNDARY CONDITIONS ON THE SURFACE

(1) For a charged conductor with surface charge density $\sigma(\vec{r})$.
Since

$$\varphi_{\text{outside}} - \varphi_{\text{inside}} = \lim_{\delta\to 0}\left(-\int_{-\delta}^{\delta} \vec{E}\cdot d\vec{s}\right) = 0$$

the potential is continuous across the surface of the conductor (Dirichlet boundary conditions).

(2) A charged conductor with surface charge density σ. We choose a Gaussian surface for an extreme small area $d\vec{a}$ and let the thickness go to zero to avoid the parallel components of $\vec{E}$ through the Gaussian surface.



Thus, we obtain

$$\oint_S \vec{E}\cdot d\vec{a} = \underbrace{E_{1\perp}da}_{\text{Air}} - \underbrace{E_{2\perp}}_{\substack{=0 \\ \text{conductor}}} da = \frac{\sigma da}{\epsilon_0} \Rightarrow \underbrace{E_{1\perp}}_{\text{Air}} - \underbrace{E_{2\perp}}_{\text{conductor}} = \frac{\sigma}{\epsilon_0}$$

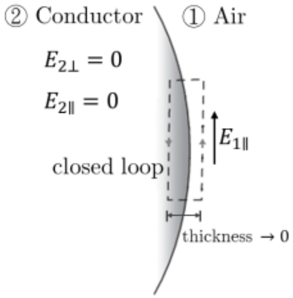
$E_\perp$ is discontinuous across the surface of the conductor by an amount σ/ϵ_0.
Since

$$E_\perp = -\nabla\varphi \cdot \hat{n} = -\frac{\partial\varphi}{\partial n}$$

the discontinuity of $E_\perp$ can be expressed in terms of the normal derivative of φ as

$$\frac{\partial\varphi_{\text{outside}}}{\partial n} - \frac{\partial\varphi_{\text{inside}}}{\partial n} = -\frac{\sigma(\vec{r})}{\epsilon_0} \cdots\cdots \text{Neumann boundary conditions}$$

(3) For if we use Stokes' theorem and let the width of the closed loop go to zero.



Thus, we obtain

$$\oint_C \vec{E} \cdot d\vec{s} = \underbrace{E_{1\parallel} ds}_{\text{Air}} - \underbrace{E_{2\parallel}}_{\substack{=0 \\ \text{conductor}}} ds = 0 \Rightarrow \underbrace{E_{1\parallel}}_{\text{Air}} - \underbrace{E_{2\parallel}}_{\text{conductor}} = 0$$

$E_\parallel$ is continuous across the surface of the conductor.

Thus, we obtain that φ is constant on the surface (Dirichlet boundary conditions).

3-2 Image Charge Method

A. METHOD OF IMAGE CHARGES

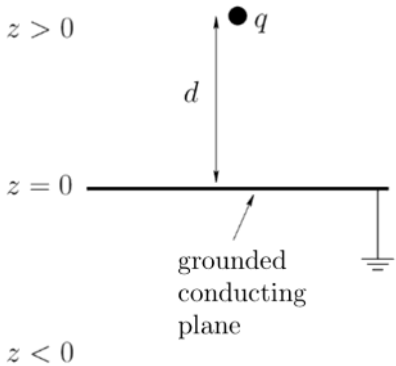
(1) The method of image charges is a calculational trick that replaces the original boundary by appropriate image charges in lieu of a formal solution of Poisson's or Laplace's equation so that the original problem is greatly simplified.

(2) The basic principle of the method of images is the uniqueness theorem. As long as (i) the solution satisfies Poisson's or Laplace's equation and (ii) the solution satisfies the given boundary condition, the simplest solution should be taken.

(3) For instance, it can be quite difficult to evaluate the charge distribution that forms on a conductor close to a charge, but we know the conductor is an equipotential surface. By adding a "false" charge outside of the region of interest that creates the same equipotential surface, and removing the conductor, we end up with the same field configuration.

B. INFINITE PLANE OF CONDUCTOR

(1) A point charge near an infinite grounded conducting plane.



- Method I
  By direct evaluation:

$$\varphi(x,y,z) = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}}_{\substack{\text{point}\\ \text{charge}}} + \underbrace{\frac{1}{4\pi\epsilon_0}\int\frac{\sigma}{r}da}_{\substack{\text{conducting}\\ \text{plane}}}$$

We don't know the surface charge distribution of the conducting plane.

- Method II

  Solving Poisson's equation

  $$\nabla^2\varphi = -\frac{\sigma}{\epsilon_0}$$

  with boundary conditions:

  (i) The conducting plate is an equipotential surface and the potential of the plate is zero,

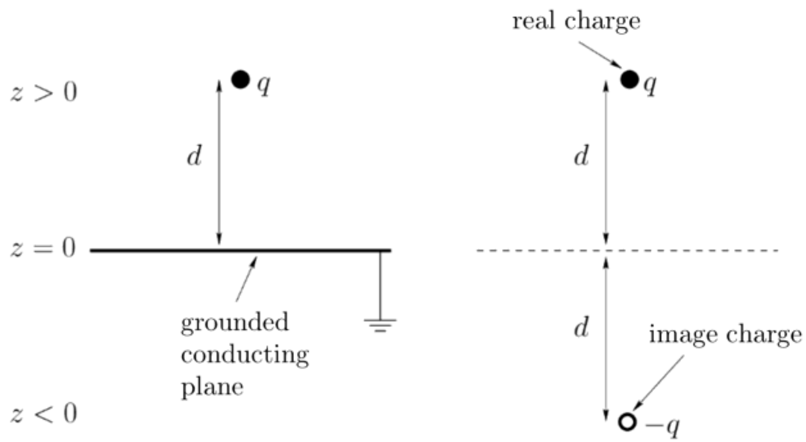
  $\varphi(0) = 0$ when $z = 0$

  (ii) The potential at infinity is zero,

  $\varphi(r) \to 0$ as $r \to \infty$

- Method III

  By method of images:



We shall refer to this as the *analog problem* and obtain

$$\varphi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

with properties:

$$\varphi(x,y,0) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(-d)^2}} - \frac{q}{\sqrt{x^2+y^2+d^2}}\right] = 0$$

$$\varphi(x,y,z) \to 0 \text{ as } x^2+y^2+z^2 \to \infty$$

Moreover, in the region $z > 0$, $\varphi(x,y,z)$ satisfies Poisson's equation for a point charge q located at coordinates $(0,0,d)$. Thus, in this region, $\varphi(x, y, z)$ is a solution to the problem. Now, the uniqueness theorem tells us that there is only one solution to Poisson's equation that satisfies a given set of boundary conditions. So, $\varphi(x, y, z)$ must be the correct potential in the region $z > 0$.

(2) Induced surface charge

The density of charge on the surface is

$$E_{\perp} = \frac{\sigma}{\epsilon_0} \Rightarrow \sigma = \epsilon_0 E_{\perp} = -\epsilon_0 \nabla\varphi$$

So

$$\begin{aligned}
\sigma &= -\epsilon_0 \frac{\partial \varphi}{\partial z} \\
&= -\frac{1}{4\pi}\left[\frac{-q(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{q(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right]_{z=0} \\
&= -\frac{1}{4\pi}\left[\frac{-q(-d)}{\left(x^2+y^2+(-d)^2\right)^{3/2}} + \frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}}\right] \\
&= -\frac{1}{2\pi}\frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}}
\end{aligned}$$

The induced charge is negative (assuming q is positive) and greatest at $x = y = 0$, i.e., the plane is closest to the point charge.

The total charge induced on the plane is

$$\begin{aligned}
Q &= \int \sigma \, da \\
&= -\frac{1}{2\pi}\int_0^{\infty} \frac{qd}{\left(x^2+y^2+d^2\right)^{3/2}} 2\pi r dr \\
&= -qd \int_0^{\infty} \frac{1}{(r^2+d^2)^{3/2}} \frac{d\left(r^2\right)}{2}, \qquad r^2 = x^2 + y^2 \\
&= qd\left[\frac{1}{(r^2+d^2)^{1/2}}\right]_0^{\infty} \\
&= -q
\end{aligned}$$

So, the total charge induced on the plate is equal and opposite to the point charge that induces it.

(3) Force and energy

The force acting on the charge at coordinates $(0,0,d)$ is

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

The potential energy:

- From method of images, with the two point charges and no conductor, we have

$$W_{\text{analog}} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d} = -\frac{q^2}{8\pi\epsilon_0 d}$$

But for a single charge and conducting plane, the energy is half of this:

$$W = \frac{1}{2}W_{\text{analog}} = -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d} = -\frac{q^2}{16\pi\epsilon_0 d}$$

- From the work required to bring q in from infinity,

$$W = \int_{\infty}^{d}\vec{F}\cdot d\vec{s} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}dz = \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)_{\infty}^{d} = -\frac{q^2}{16\pi\epsilon_0 d}$$

- From the energy stored in the surrounding electric field,

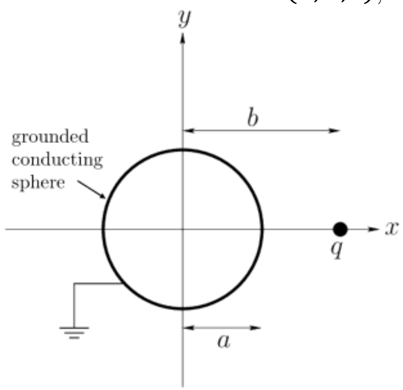
$$W = \frac{\epsilon_0}{2}\int_{z>0} E^2\, d\tau$$

Since $E^2(x,y,-z) = E^2(x,y,z)$, we get

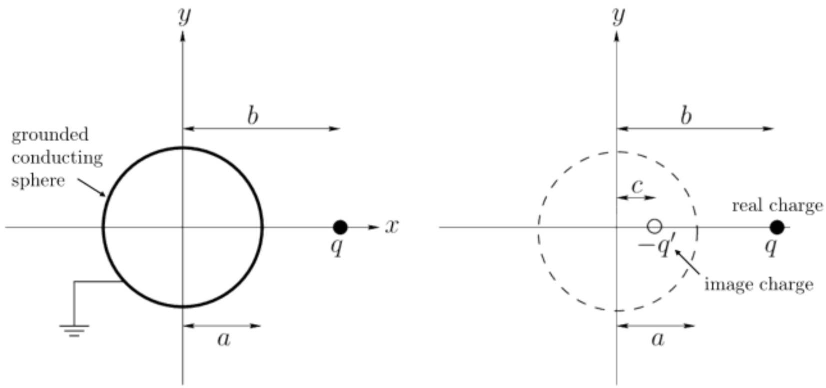
$$W_{\text{analog}} = 2\frac{\epsilon_0}{2}\int_{z>0} E^2\, d\tau = 2W \Rightarrow W = \frac{1}{2}W_{\text{analog}} = -\frac{q^2}{16\pi\epsilon_0 d}$$

C. SPHERE OF GROUNDED CONDUCTOR

(1) A grounded conducting sphere of radius a centered on the origin. Suppose that a point electric charge q is placed outside the sphere at Cartesian coordinates $(b,0,0)$, where $b > a$.



By method of images:



We shall refer to this as the *analog problem* and obtain

$$\varphi(x,y,z)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-b)^2+y^2+z^2}}-\frac{q'}{\sqrt{(x-c)^2+y^2+z^2}}\right]$$

Now, the image charge must be chosen so as to make the surface $\varphi=0$ correspond to the surface of the sphere.

$$\frac{q}{\sqrt{(x-b)^2+y^2+z^2}}=\frac{q'}{\sqrt{(x-c)^2+y^2+z^2}}$$

$$\Rightarrow \frac{q^2}{(x-b)^2+y^2+z^2}=\frac{q'^2}{(x-c)^2+y^2+z^2}$$

Let $\lambda=q'^2/q^2$. We obtain

$$(x-c)^2+y^2+z^2=\lambda\left[(x-b)^2+y^2+z^2\right]$$

$$\Rightarrow (1-\lambda)x^2+(1-\lambda)y^2+(1-\lambda)z^2-2(c-\lambda b)x+c^2-\lambda b^2=0$$

$$\Rightarrow x^2+\frac{2(c-\lambda b)}{\lambda-1}x+y^2+z^2=\frac{c^2-\lambda b^2}{\lambda-1}$$

Since

$$x^2+y^2+z^2=a^2$$

we have

$$a^2=\frac{c^2-\lambda b^2}{\lambda-1}=\frac{c^2-\frac{c}{b}b^2}{\frac{c}{b}-1}=\frac{c(c-b)}{\frac{c-b}{b}}=bc\Rightarrow c=\frac{a^2}{b}$$

$$\lambda=\frac{c}{b}=\frac{a^2}{b^2}$$

Thus, we obtain

$$q'=\sqrt{\lambda}q=\sqrt{\frac{a^2}{b^2}}q=\frac{a}{b}q$$

The potential is

$$\varphi(x,y,z)=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-b)^2+y^2+z^2}}-\frac{\frac{a}{b}q}{\sqrt{\left(x-\frac{a^2}{b}\right)^2+y^2+z^2}}\right]$$

(2) The net charge induced on the surface of the conducting sphere is

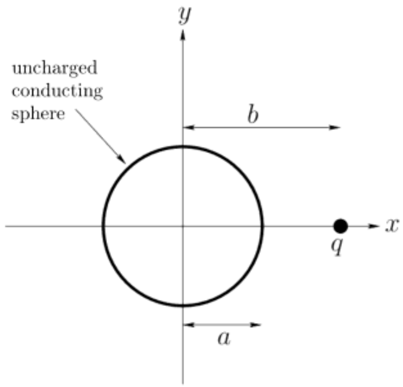
$$Q=-q'=-\frac{a}{b}q$$

(3) The force of attraction between the sphere and the original charge is

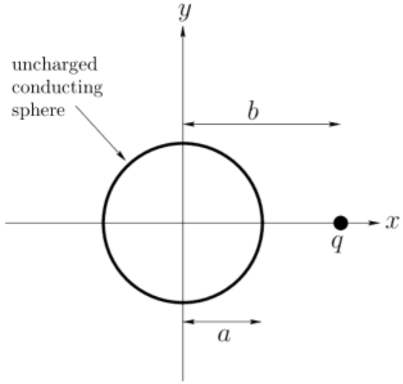
$$\vec{F}=-\frac{1}{4\pi\epsilon_0}\frac{qq'}{(b-c)^2}\hat{x}=-\frac{1}{4\pi\epsilon_0}\frac{\frac{a}{b}q^2}{\left(b-\frac{a^2}{b}\right)^2}\hat{x}=-\frac{q^2}{4\pi\epsilon_0}\frac{ab}{(b^2-a^2)^2}\hat{x}$$

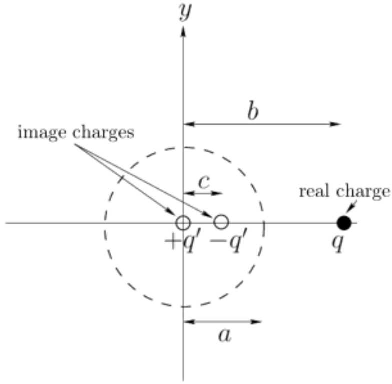
D. SPHERE OF UNCHARGED CONDUCTOR

(1) An insulated, uncharged, conducting sphere of radius a, centered on the origin, in the presence of a point electric charge q placed outside the sphere at Cartesian coordinates $(b,0,0)$, where $b>a$.



By method of images:





We shall refer to this as the *analog problem* and obtain

$$\varphi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-b)^2+y^2+z^2}} + \frac{q'}{\sqrt{x^2+y^2+z^2}} - \frac{q'}{\sqrt{(x-c)^2+y^2+z^2}}\right]$$

Now, the image charge must be chosen so as to make the surface

$$\varphi = \frac{q'}{4\pi\epsilon_0 a}$$

correspond to the surface of the sphere. Thus, we have

$$q' = \frac{a}{b}q$$

$$c = \frac{a^2}{b}$$

The potential is

$$\varphi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{(x-b)^2+y^2+z^2}} + \frac{\frac{a}{b}q}{\sqrt{x^2+y^2+z^2}} - \frac{\frac{a}{b}q}{\sqrt{\left(x-\frac{a^2}{b}\right)^2+y^2+z^2}}\right]$$

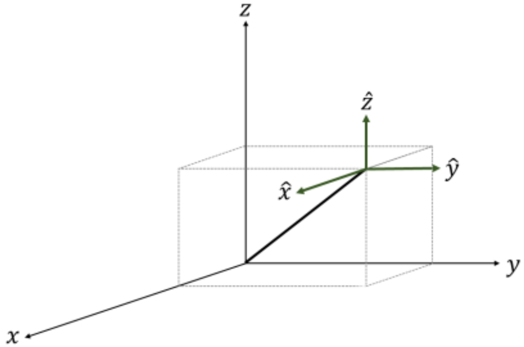
(2) The force of attraction between the sphere and the original charge is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(b-c)^2}\hat{x} - \frac{1}{4\pi\epsilon_0}\frac{qq'}{b^2}\hat{x} = \frac{q^2}{4\pi\epsilon_0}\left(\frac{a}{b}\right)^3\frac{(2b^2-a^2)}{(b^2-a^2)^2}\hat{x}$$

3-3 Separation of Variables

A. SYMMETRY AND CURVILINEAR COORDINATES

(1) Cartesian coordinates



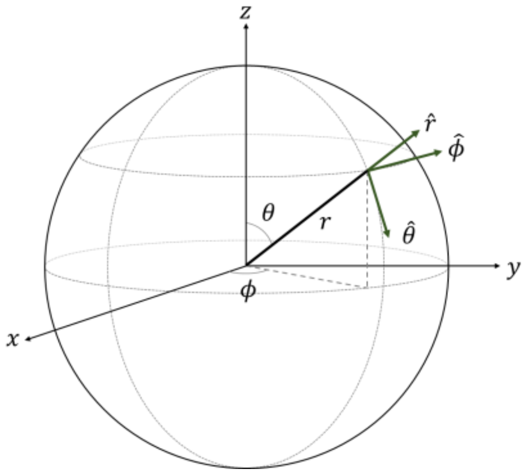
The potential and electric field:

$$\varphi = \varphi(x, y, z), \qquad \vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

Laplace's equation reads:

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

(2) Spherical symmetry and coordinates



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

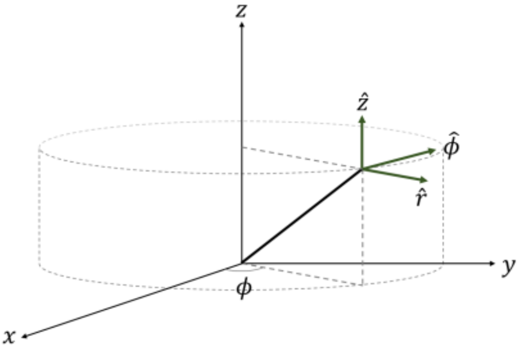
The potential and electric field:

$$\varphi = \varphi(r, \theta, \phi), \qquad \vec{E} = E_r\hat{r} + E_\theta\hat{\theta} + E_\phi\hat{\phi}$$

Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} = 0$$

(3) Cylindrical symmetry and coordinates



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

The potential and electric field:

$$\varphi = \varphi(r, \phi, z), \qquad \vec{E} = E_r\hat{r} + E_\phi\hat{\phi} + E_z\hat{z}$$

Laplace's equation reads:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

B. SOLUTIONS IN CARTESIAN COORDINATES

(1) For rectangular objects, Laplace's equation reads

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

Assume φ is in the form of products:

$$\varphi = X(x)Y(y)Z(z)$$

Laplace's equation becomes

$$\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2} = 0$$

(2) Dividing by $\varphi = X(x)Y(y)Z(z)$, thus, we have

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

Since the first term depends only on x, the second only on y, and the third on z, it follows that each must be a constant:

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = k^2 + l^2$$

$$\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = -k^2$$

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = -l^2$$

Then we obtain three ordinary differential equations

The X equation:

$$\frac{d^2 X(x)}{dx^2} - (k^2 + l^2)X(x) = 0$$

The Y equation:

$$\frac{d^2 Y(y)}{dy^2} + k^2 Y(y) = 0$$

The Z equation:

$$\frac{d^2 Z(z)}{dz^2} + l^2 Z(z) = 0$$

(3) The general solution of the X equation

$$X(x) = Ae^{\sqrt{k^2+}l^2 x} + Be^{-\sqrt{k^2+}l^2 x}$$

The general solution of the Y equation

$$Y(y) = C\sin ky + D\cos ky$$

The general solution of the Y equation

$$Z(z) = E\sin lz + F\cos lz$$

(4) Since separation of variables yields an infinite set of solutions, one for each k and l, the general solution is the linear combination of separable solutions:

$$\varphi(x,y,z) = \sum_{k,l}^{\infty}\left(Ae^{\sqrt{k^2+}l^2 x} + Be^{-\sqrt{k^2+}l^2 x}\right)(C\sin ky + D\cos ky) \times (E\sin lz + F\cos lz)$$

C. SOLUTIONS IN SPHERICAL COORDINATES

(1) For round objects, spherical coordinates are more natural and Laplace's equation reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}=0$$

Assume φ is in the form of products,

$$\varphi=R(r)\Theta(\theta)\Phi(\phi)$$

Laplace's equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)\Theta(\theta)\Phi(\phi)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R(r)\Theta(\theta)\Phi(\phi)}{\partial\theta}\right)$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2R(r)\Theta(\theta)\Phi(\phi)}{\partial\phi^2}=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)\Theta(\theta)\Phi(\phi)}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R(r)\Theta(\theta)\Phi(\phi)}{\partial\theta}\right)$$
$$+\frac{1}{\sin^2\theta}\frac{\partial^2R(r)\Theta(\theta)\Phi(\phi)}{\partial\phi^2}=0$$

(2) Dividing by $\varphi=R(r)\Theta(\theta)\Phi(\phi)$, thus, we have

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)$$
$$+\frac{1}{\Phi(\phi)}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0$$

Each term therefore must be a constant:

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)=l(l+1)$$
$$\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\Phi(\phi)}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-l(l+1)$$

The second equation gives

$$\frac{1}{\Theta(\theta)}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+l(l+1)\sin^2\theta+\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=0$$

Then, each term must be a constant:

$$\frac{1}{\Theta(\theta)}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+l(l+1)\sin^2\theta=m$$

$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = -m$$

Thus we obtain three ordinary differential equations

The radial equation:

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - l(l+1)R(r) = 0$$

The angular equation:

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) + l(l+1)\sin^2\theta\,\Theta(\theta) - m\Theta(\theta) = 0$$

The azimuthal equation:

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + m\Phi(\phi) = 0$$

(3) The general solution of the radial equation

$$R(r) = A_l r^l + \frac{B_l}{r^{l+1}}, \qquad l = 0,1,2,3\cdots$$

The general solution of the angular equation

$\Theta(\theta) = P_{l,m}(\cos\theta)$ ····· associated Legendre polynomials

The general solution of the azimuthal equation

$$\Phi(\phi) = Ce^{im\phi}$$

Thus, the most general separable solution to Laplace's equation is,

$$\varphi_l = R_l(r)\Theta_{l,m}(\theta)\Phi_m(\phi) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_{l,m}(\cos\theta)e^{im\phi}$$

Since separation of variables yields an infinite set of solutions, one for each l, the general solution is the linear combination of separable solutions:

$$\varphi = \sum_{\substack{l=0\\m=-l,l}}^{\infty} R_l(r)\Theta_{l,m}(\theta)\Phi_m(\phi)$$

$$= \sum_{\substack{l=0\\m=-l,l}}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_{l,m}(\cos\theta)e^{im\phi}$$

D. SOLUTIONS IN CYLINDRICAL COORDINATES

(1) For cylinders, cylindrical coordinates are more natural and Laplace's equation reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \phi^2}+\frac{\partial^2 \varphi}{\partial z^2}=0$$

Assume φ is in the form of products,

$$\varphi = R(r)\Phi(\phi)Z(z)$$

Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)\Phi(\phi)Z(z)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 R(r)\Phi(\phi)Z(z)}{\partial \phi^2}$$
$$+\frac{\partial^2 R(r)\Phi(\phi)Z(z)}{\partial z^2}=0$$

(2) Dividing by $\varphi = R(r)\Phi(\phi)Z(z)$, thus, we have

$$\frac{1}{R(r)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Phi(\phi)}\frac{1}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}=0$$

Each term therefore must be a constant:

$$\frac{1}{R(r)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)+\frac{1}{\Phi(\phi)}\frac{1}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}=k^2$$

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}=-k^2$$

The first equation gives

$$\frac{1}{R(r)}r\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)-k^2r^2+\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}=0$$

Then, each term must be a constant:

$$\frac{1}{R(r)}r\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right)-k^2r^2=l^2$$

$$\frac{1}{\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2}=-l^2$$

Thus we obtain three ordinary differential equations

The radial equation:

$$r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)-k^2r^2R(r)-l^2R(r)=0$$

The azimuthal equation:

$$\frac{d^2\Phi(\phi)}{d\phi^2}+l^2\Phi(\phi)=0$$

The Z equation:

$$\frac{d^2Z(z)}{dz^2}+k^2Z(z)=0$$

E. EXAMPLES:

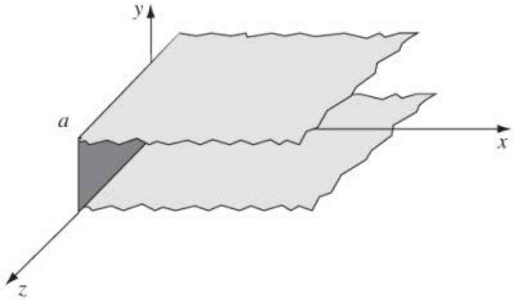
(1) Two infinite grounded metal plates lie parallel to the xz plane, subject to the boundary conditions:

$\varphi = 0$ when $y = 0$

$\varphi = 0$ when $y = a$

$\varphi \to 0$ as $x \to \infty$

$\varphi = \varphi_0$ when $x = 0 \cdots\cdots$ (iv)



Find the potential inside this "slot."

ANSWER:

Laplace's equation is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

Using separation of variables, we obtain two ordinary differential equations:

The X equation:

$$\frac{d^2 X(x)}{dx^2} - k^2 X(x) = 0$$

The Y equation:

$$\frac{d^2 Y(y)}{dy^2} + k^2 Y(y) = 0$$

The solution is

$$\varphi_k = X_k(x) Y_k(y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C \sin ky + D \cos ky\right)$$

Since

$\varphi = 0$ when $y = 0 \Rightarrow C \cdot 0 + D = 0 \Rightarrow D = 0$

$\varphi = 0$ when $y = a \Rightarrow C \sin ka = 0 \Rightarrow k_n = \frac{n\pi}{a}$, $n = 1,2,3 \cdots$

$\varphi \to 0$ as $x \to \infty \Rightarrow Ae^{k\infty} = 0 \Rightarrow A = 0$

The general solution is

$$\varphi(x,y)=\sum_{n=1}^{\infty} C_n e^{-k_n x}\sin k_n y\,,\qquad k_n=\frac{n\pi}{a}$$

This solution meets all the boundary conditions except (iv).

At $x=0$, we require that

$$\varphi_0=\sum_{n=1}^{\infty} C_n \sin k_n y$$

Using the orthogonality relation:

$$\int_0^a \sin k_{n'}y \sin k_n y\, dy=\frac{1}{2}\int_0^a \left[\cos(k_{n'}-k_n)y-\cos(k_{n'}+k_n)y\right]dy$$

$$=\begin{cases}\dfrac{a}{2}, & n=n'\\ 0, & n\neq n'\end{cases}$$

$$\Rightarrow\quad \underbrace{\frac{2}{a}}_{\substack{\text{normalization}\\ \text{constant}}}\int_0^a \sin k_{n'}y\sin k_n y\,dy=\delta_{n,n'}=\begin{cases}1, & n=n'\\ 0, & n\neq n'\end{cases}$$

we have

$$\frac{2}{a}\int_0^a \sin k_{n'}y\,\varphi_0 dy=\frac{2}{a}\int_0^a \sin k_{n'}y\sum_{n=1}^{\infty} C_n\sin k_n y\,dy$$

$$\Rightarrow \frac{2}{a}\int_0^a \sin k_{n'}y\,\varphi_0 dy=\sum_{n=1}^{\infty} C_n\frac{2}{a}\int_0^a \sin k_{n'}y\sin k_n y\,dy$$

$$\Rightarrow \frac{2}{a}\int_0^a \sin k_{n'}y\,\varphi_0 dy=\sum_{n=1}^{\infty} C_n\,\delta_{n,n'}$$

Thus, we obtain C_n as

$$C_{n'}=\frac{2}{a}\int_0^a \varphi_0\sin\left(\frac{n'\pi y}{a}\right)dy$$

$$=\frac{2\varphi_0}{a}\frac{a}{n'\pi}(1-\cos n'\pi)$$

$$=\begin{cases}0\quad, & n' \text{ is even}\\ \dfrac{4\varphi_0}{n'\pi}\,, & n' \text{ is odd}\end{cases}$$

Therefore, the potential is

$$\varphi(x,y)=\frac{4\varphi_0}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n}e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)=\frac{2\varphi_0}{\pi}\tan^{-1}\left(\frac{\sin\frac{\pi y}{a}}{\sinh\frac{\pi x}{a}}\right)$$

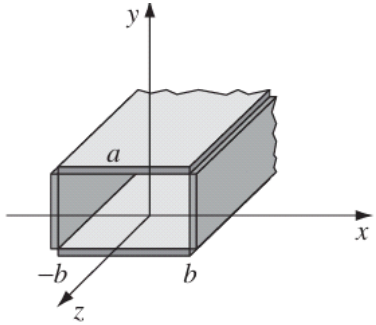
(2) Two infinitely-long grounded metal plates are connected by metal strips maintained at a constant potential φ_0, subject to the boundary conditions:

$\varphi = 0$ when $y = 0$

$\varphi = 0$ when $y = a$

$\varphi = \varphi_0$ when $x = b$ ····· (iii)

$\varphi = \varphi_0$ when $x = -b$ ····· (iv)



Find the potential inside the resulting rectangular pipe.

ANSWER:

Laplace's equation is

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

The solution is

$$\varphi_k = X_k(x)Y_k(y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin ky + D\cos ky\right)$$

The boundary conditions are

$$\varphi = 0 \text{ when } y = 0 \Rightarrow C\cdot 0 + D = 0 \Rightarrow D = 0$$

$$\varphi = 0 \text{ when } y = a \Rightarrow C\sin ka = 0 \Rightarrow k_n = \frac{n\pi}{a},\ n = 1,2,3\cdots$$

Combining the remaining constants, we are left with

$$\varphi_k = X_k(x)Y_k(y) = \left(Ae^{kx} + Be^{-kx}\right)C\sin k_n y$$

This solution meets all the boundary conditions except (iii) and (iv). The situation is symmetric with respect to x, so $\varphi(-x,y) = \varphi(x,y)$, and it follows that $A = B$. Using

$$e^{kx} + e^{-kx} = 2\cosh k_n x$$

Thus, the potential becomes

$$\varphi(x,y) = \sum_{n=1}^{\infty} C_n \cosh k_n x \sin k_n y, \qquad k_n = \frac{n\pi}{a}$$

At $x = b$, we require that

$$\varphi_0 = \sum_{n=1}^{\infty} C_n \cosh k_n b \sin k_n y$$

Using the orthogonality relation, we have

$$\frac{2}{a}\int_0^a \sin k_{n'}y\, \varphi_0 dy = \frac{2}{a}\int_0^a \sin k_{n'}y \sum_{n=1}^{\infty} C_n \cosh k_n b \sin k_n y\, dy$$

$$\Rightarrow \frac{2}{a}\int_0^a \sin k_{n'}y\, \varphi_0 dy = \sum_{n=1}^{\infty} C_n \cosh k_n b \frac{2}{a}\int_0^a \sin k_{n'}y \sin k_n y\, dy$$

$$\Rightarrow \frac{2}{a}\int_0^a \sin k_{n'}y\, \varphi_0 dy = \sum_{n=1}^{\infty} C_n \cosh k_n b\, \delta_{n,n'}$$

Thus, we obtain

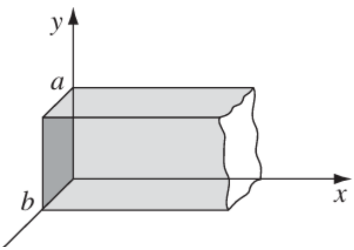
$$\begin{aligned} C_{n'} \cosh k_{n'} b &= \frac{2\varphi_0}{a}\int_0^a \sin\left(\frac{n'\pi y}{a}\right) dy \\ &= \frac{2\varphi_0}{a}\frac{a}{n'\pi}(1-\cos n'\pi) \\ &= \begin{cases} 0 \quad , & n' \text{ is even} \\ \dfrac{4\varphi_0}{n'\pi} \;, & n' \text{ is odd} \end{cases} \end{aligned}$$

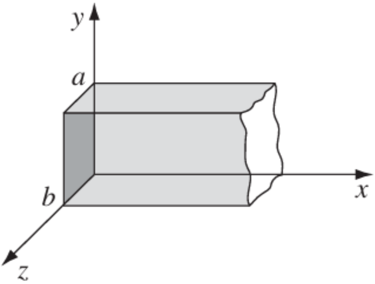
$$\Rightarrow C_{n'} = \frac{4\varphi_0}{n'\pi}\frac{1}{\cosh k_{n'} b}$$

Therefore, the potential is

$$\begin{aligned} \varphi(x,y) &= \sum_{n=1,3,5,\cdots}^{\infty} \frac{4\varphi_0}{\pi n}\frac{1}{\cosh k_n b} \cosh k_n x \sin(k_n y) \\ &= \frac{4\varphi_0}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n}\frac{\cosh \frac{n\pi}{a}x}{\cosh \frac{n\pi}{a}b} \sin\left(\frac{n\pi}{a}y\right) \end{aligned}$$

(3) An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential φ_0. Find the potential inside the pipe.





ANSWER:

The boundary conditions are

$\varphi = 0$ when $y = 0$

$\varphi = 0$ when $y = a$

$\varphi = 0$ when $z = 0$

$\varphi = 0$ when $z = a$

$\varphi \to 0$ when $x \to \infty$

$\varphi = \varphi_0$ when $x = 0 \cdots\cdots$ (vi)

The general solution of the X equation

$$X(x) = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}$$

The general solution of the Y equation

$$Y(y) = C \sin ky + D \cos ky$$

The general solution of the Y equation

$$Z(z) = E \sin lz + F \cos lz$$

The boundary conditions are

$\varphi = 0$ when $y = 0 \Rightarrow C \cdot 0 + D = 0 \Rightarrow D = 0$

$\varphi = 0$ when $z = 0 \Rightarrow E \cdot 0 + F = 0 \Rightarrow F = 0$

$\varphi = 0$ when $y = a \Rightarrow C \sin ka = 0 \Rightarrow k_n = \frac{n\pi}{a},\ n = 1,2,3 \cdots$

$\varphi = 0$ when $z = a \Rightarrow E \sin la = 0 \Rightarrow l_m = \frac{m\pi}{a},\ m = 1,2,3 \cdots$

$\varphi \to 0$ when $x \to \infty \Rightarrow Ae^{\sqrt{k^2+l^2}\infty} = 0 \Rightarrow A = 0$

Combining the remaining constants, we are left with

$$\varphi(x, y, z) = \sum_{\substack{n=1 \\ m=1}}^{\infty} C_{nm} e^{-\sqrt{k_n^2+l_m^2}x} \sin k_n y \sin l_m z$$

This solution meets all the boundary conditions except (vi).

At $x = 0$, we require that

$$\varphi_0 = \sum_{\substack{n=1 \\ m=1}}^{\infty} C_{nm} \sin k_n y \sin l_m z$$

Using the orthogonality relation, we have

$$\frac{2}{a}\int_0^a \sin k_{n'}y \frac{2}{b}\int_0^b \sin l_{m'}z\, \varphi_0 dydz$$

$$= \frac{2}{a}\int_0^a \sin k_{n'}y \frac{2}{b}\int_0^b \sin l_{m'}y \sum_{\substack{n=1\\m=1}}^{\infty} C_{nm}\sin k_n y \sin l_m z\, dydz$$

$$\Rightarrow \frac{4}{ab}\int_0^a\int_0^b \sin k_{n'}y \sin l_{m'}z\, \varphi_0 dydz$$

$$= \sum_{\substack{n=1\\m=1}}^{\infty} C_{nm}\frac{2}{a}\int_0^a \sin k_{n'}y \sin k_n y\, dy \frac{2}{b}\int_0^b \sin l_{m'}z \sin l_m z\, dz$$

$$\Rightarrow \frac{4}{ab}\int_0^a\int_0^b \sin k_{n'}y \sin l_{m'}z\, \varphi_0 dydz = \sum_{\substack{n=1\\m=1}}^{\infty} C_{nm}\,\delta_{n',n}\delta_{m',m}$$

Thus, we obtain

$$C_{n',m'} = \frac{4}{ab}\int_0^a\int_0^b \sin k_{n'}y \sin l_{m'}z\, \varphi_0 dydz$$

$$= \frac{4\varphi_0}{ab}\int_0^a\int_0^b \sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)dydz$$

$$= \begin{cases} 0 \quad , & n' \text{ or } m' \text{ is even} \\ \dfrac{16\varphi_0}{n'm'\pi^2}, & n' \text{ and } m' \text{ is odd} \end{cases}$$

Therefore, the potential is

$$\varphi(x,y) = \sum_{\substack{n=1,3,5,\cdots\\m=1,3,5,\cdots}}^{\infty} \frac{16\varphi_0}{nm\pi^2} e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}x}\sin k_n y \sin l_m z$$

$$= \frac{16\varphi_0}{\pi^2}\sum_{\substack{n=1,3,5,\cdots\\m=1,3,5,\cdots}}^{\infty} \frac{1}{nm} e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}x}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

(4) The potential $\varphi_0(\theta) = k\sin^2\theta/2$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside and outside the sphere.

ANSWER:

Assume, in the case of azimuthal symmetry, $\varphi = R(r)\Theta(\theta)$, we have two ordinary differential equations

The radial equation:

$$\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)-l(l+1)R(r)=0$$

The angular equation:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+l(l+1)\sin\theta\,\Theta(\theta)=0$$

The general solution of the radial equation

$$R(r)=A_l r^l+\frac{B_l}{r^{l+1}}$$

The general solution of the angular equation

$\Theta(\theta)=P_l(\cos\theta)$ ⋯⋯ Legendre polynomials

where

$$P_0=1$$
$$P_1=\cos\theta$$
$$P_2=\frac{(3\cos^2\theta-1)}{2}$$
$$P_3=\frac{(5\cos^3\theta-3\cos\theta)}{2}$$
$$P_4=\frac{(35\cos^4\theta-30\cos^2\theta+3)}{8}$$

Thus, in the case of azimuthal symmetry, the most general separable solution to Laplace's equation is,

$$\varphi_l=R_l(r)\Theta_l(\theta)=\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Since separation of variables yields an infinite set of solutions, one for each l, the general solution is the linear combination of separable solutions:

$$\varphi=\sum_{l=0}^{\infty}R_l(r)\Theta_l(\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

- Inside the sphere:

$B_l=0$ for all l

Thus,

$$\varphi(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$

At $r=R$, we require that

$$\varphi_0(\theta)=\sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)$$

Using the orthogonality relation:

$$\underbrace{\frac{2l+1}{2}}_{\substack{\text{normalization}\\ \text{constant}}} \int_0^\pi P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta = \delta_{l,l'} = \begin{cases} 0, & l' \neq l \\ 1, & l' = l \end{cases}$$

we have

$$\frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta)\varphi_0(\theta)\sin\theta\, d\theta$$

$$= \frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta)\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \frac{2l'+1}{2}\int_0^\pi \varphi_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \sum_{l=0}^{\infty} A_l R^l \frac{2l'+1}{2}\int_0^\pi P_{l'}(\cos\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \frac{2l'+1}{2}\int_0^\pi \varphi_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta = \sum_{l=0}^{\infty} A_l R^l \,\delta_{l,l'}$$

$$\Rightarrow \frac{2l'+1}{2}\int_0^\pi \varphi_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta = A_{l'} R^{l'}$$

Thus, we obtain

$$A_{l'} = \frac{1}{R^{l'}}\cdot\frac{2l'+1}{2}\int_0^\pi \varphi_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{1}{R^{l'}}\cdot\frac{2l'+1}{2}\int_0^\pi k\sin^2\left(\frac{\theta}{2}\right) P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{1}{R^{l'}}\cdot\frac{2l'+1}{2}\int_0^\pi \frac{k}{2}(1-\cos\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{k}{2}\frac{1}{R^{l'}}\cdot\frac{2l'+1}{2}\int_0^\pi (P_0 - P_1) P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{k}{2}\frac{1}{R^{l'}}\left(\delta_{0,l'} - \delta_{1,l'}\right)$$

$$\Rightarrow \begin{cases} A_0 = \dfrac{k}{2}\dfrac{1}{R^0} = \dfrac{k}{2} \quad , & l' = 0 \\ A_1 = -\dfrac{k}{2}\dfrac{1}{R^1} = -\dfrac{k}{2R}, & l' = 1 \end{cases}$$

Therefore, the potential is

$$\varphi = A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

- Outside the sphere:

  $A_l = 0$ for all l

  Thus,

  $$\varphi(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

  At the surface of the sphere, we require that

  $$\varphi_0 = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

  Using the orthogonality relation, we have

  $$\frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)\varphi_0(\theta)\sin\theta\,d\theta$$
  $$= \frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)\sin\theta\,d\theta$$

  $$\Rightarrow \frac{2l'+1}{2}\int_0^{\pi} \varphi_0(\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta$$
  $$= \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\,d\theta$$

  $$\Rightarrow \frac{2l'+1}{2}\int_0^{\pi} \varphi_0(\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}\delta_{l,l'}$$

  $$\Rightarrow \frac{2l'+1}{2}\int_0^{\pi} \varphi_0(\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta = \frac{B_{l'}}{R^{l'+1}}$$

  Thus, we obtain

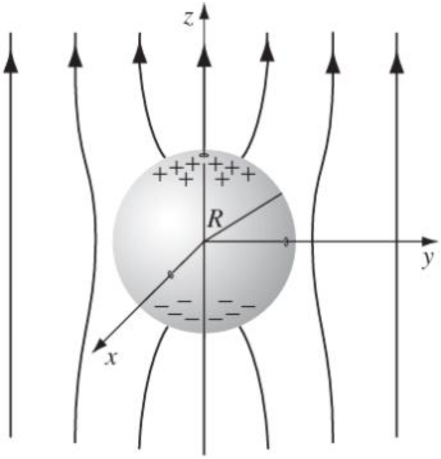
  $$B_{l'} = R^{l'+1}\cdot\frac{2l'+1}{2}\int_0^{\pi} \varphi_0(\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta$$
  $$= R^{l'+1}\cdot\frac{2l'+1}{2}\int_0^{\pi} k\sin^2\left(\frac{\theta}{2}\right)P_{l'}(\cos\theta)\sin\theta\,d\theta$$
  $$= R^{l'+1}\cdot\frac{2l'+1}{2}\int_0^{\pi} \frac{k}{2}(1-\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta$$
  $$= \frac{k}{2}R^{l'+1}\cdot\frac{2l'+1}{2}\int_0^{\pi}\left(P_0 - P_1\right)P_{l'}(\cos\theta)\sin\theta\,d\theta$$
  $$= \frac{k}{2}R^{l'+1}\left(\delta_{0,l'} - \delta_{1,l'}\right)$$

$$\Rightarrow \begin{cases} B_0 = \frac{k}{2}R^{0+1} = \frac{k}{2}R \quad , & l' = 0 \\ B_1 = -\frac{k}{2}R^{1+1} = -\frac{k}{2}R^2, & l' = 1 \end{cases}$$

Therefore, the potential is

$$\varphi = \frac{B_0}{r^1}P_0(\cos\theta) + \frac{B_1}{r^2}P_1(\cos\theta) = \frac{k}{2}\left(\frac{R}{r} - \frac{R^2}{r^2}\cos\theta\right)$$

(5) An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\vec{E} = E_0\hat{z}$. The field will push positive charge to the "northern" surface of the sphere, and—symmetrically—negative charge to the "southern" surface. This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.



ANSWER:

The boundary conditions:

$$\varphi = 0 \text{ when } r = R$$

$$\varphi \to -E_0 r\cos\theta \text{ for } r \gg R$$

The potential is

$$\varphi = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = -A_l R^{2l+1}$$

So, we have

$$\varphi = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta)$$

For $r \gg R$, we have

$$-E_0 r \cos\theta = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta) \approx \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Only $l = 1$ term survive. Thus, we get

$$-E_0 r \cos\theta = A_1 r^1 P_1(\cos\theta) \Rightarrow A_1 = -E_0$$

The potential is

$$\varphi = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}} \right) P_l(\cos\theta) = -E_0 \left(r^1 - \frac{R^3}{r^2} \right) \cos\theta$$

The first term is due to the external field; the second term is attributed to the induced charge.

The induced charge density is

$$\sigma(\theta) = -\epsilon_0 \frac{\partial\varphi}{\partial r} = \epsilon_0 E_0 \left(1 + \frac{2R^3}{r^3} \right)_{r=R} \cos\theta = 3\epsilon_0 E_0 \cos\theta$$

(6) A specified charge density $\sigma(\theta) = k\cos\theta$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

ANSWER:

Inside the sphere:

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Outside the sphere:

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

The boundary conditions are

(i) at the surface of the sphere, the potential is continuous

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \Rightarrow A_l R^l = \frac{B_l}{R^{l+1}}$$

(ii) the radial derivative of φ suffers a discontinuity

$$\frac{\partial\varphi_{\text{out}}}{\partial r} - \frac{\partial\varphi_{\text{in}}}{\partial r} = -\frac{\sigma_0(\theta)}{\epsilon_0}$$

$$\Rightarrow -\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta) - \sum_{l=0}^{\infty} lA_l R^{l-1}P_l(\cos\theta) = -\frac{\sigma_0(\theta)}{\epsilon_0}$$

$$\Rightarrow \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta) = \frac{\sigma_0(\theta)}{\epsilon_0}$$

Using the orthogonality relation,

$$\frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\, d\theta = \delta_{l,l'} = \begin{cases} 0, & \text{for } l \neq l' \\ 1, & \text{for } l = l' \end{cases}$$

we have

$$\frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}P_l(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)\frac{\sigma_0(\theta)}{\epsilon_0}\sin\theta\, d\theta$$

$$\Rightarrow \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\frac{2l'+1}{2}\int_0^{\pi} P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{2l'+1}{2}\int_0^{\pi}\frac{\sigma_0(\theta)}{\epsilon_0}P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1}\,\delta_{l,l'} = \frac{2l'+1}{2}\int_0^{\pi}\frac{\sigma_0(\theta)}{\epsilon_0}P_{l'}(\cos\theta)\sin\theta\, d\theta$$

Thus, we can obtain A_l as

$$(2l'+1)A_{l'}R^{l'-1} = \frac{2l'+1}{2}\int_0^{\pi}\frac{\sigma_0(\theta)}{\epsilon_0}P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow A_{l'} = \frac{1}{\epsilon_0 R^{l'-1}}\cdot\frac{1}{2}\int_0^{\pi}\sigma_0(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{1}{\epsilon_0 R^{l'-1}}\cdot\frac{1}{2}\int_0^{\pi} k\cos\theta\, P_{l'}(\cos\theta)\sin\theta\, d\theta$$

Since $P_1(\cos\theta) = \cos\theta$, we have

$$A_{l'} = \frac{k}{\epsilon_0 R^{l'-1}(2l'+1)}\cdot\frac{2l'+1}{2}\int_0^{\pi} P_1(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{k}{\epsilon_0 R^{l'-1}(2l'+1)}\cdot\delta_{1,l'}$$

$$\Rightarrow A_{l'} = \begin{cases} 0, & \text{for } l' \neq 1 \\ \dfrac{k}{\epsilon_0 R^{1-1}(2\cdot 1+1)} = \dfrac{k}{3\epsilon_0}, & \text{for } l' = 1 \end{cases}$$

The potential inside the sphere is therefore

$$\varphi(r,\theta) = A_1 r^1 P_1(\cos\theta) = \frac{k}{3\epsilon_0} r\cos\theta$$

whereas outside the sphere

$$\varphi(r,\theta) = \frac{B_1}{r^2} P_1(\cos\theta) = \frac{A_1 R^{1+2}}{r^2} P_1(\cos\theta) = \frac{k}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta$$

❖ In particular, if $k = 3\epsilon_0 E_0$, then the potential inside is

$$\varphi(r,\theta) = \frac{3\epsilon_0 E_0}{3\epsilon_0} r\cos\theta = E_0 r\cos\theta = E_0 z$$

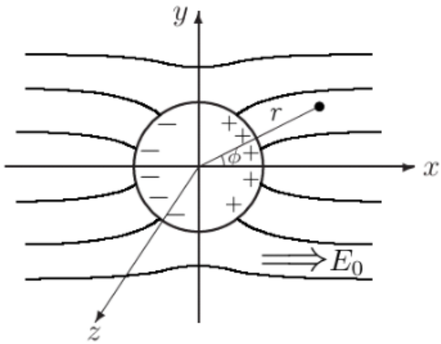
and the field is

$$\vec{E}_{\text{in}} = -E_0\hat{z}$$

which is exactly right to cancel off the external field. Outside the sphere the potential due to this surface charge is

$$\varphi(r,\theta) = \frac{3\epsilon_0 E_0}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta = E_0\frac{R^3}{r^2}\cos\theta$$

(7) Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\vec{E}_0$. Find the surface charge induced on the pipe.



ANSWER:

Assume, in the case of cylindrical symmetry, $\varphi = R(r)\Phi(\phi)$, we have two ordinary differential equations

The radial equation:

$$r\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right) - l^2 R(r) = 0$$

The azimuthal equation:

$$\frac{d^2\Phi(\phi)}{d\phi^2} + l^2\Phi(\phi) = 0$$

The general solution of the azimuthal equation

$$\Phi(\phi) = C_l \sin l\phi + D_l \cos l\phi$$

Moreover, since $\Phi(\phi + 2\pi) = \Phi(\phi)$, l must be an integer: $l = 0,1,2,3\cdots$.

The general solution of the radial equation

$$R(r) = \begin{cases} A_l r^l + \dfrac{B_l}{r^l} \quad , & l \neq 0 \\ A_0 + B_0 \ln r \, , & l = 0 \end{cases}$$

Since separation of variables yields an infinite set of solutions, one for each l, the general solution is the linear combination of separable solutions:

$$\varphi = A_0 + B_0 \ln r + \sum_{l=1}^{\infty} \left(A_l r^l + \frac{B_l}{r^l} \right) (C_l \sin l\phi + D_l \cos l\phi)$$

The boundary conditions:

$$\varphi \to -E_0 x = -E_0 r \cos\phi \text{ for } r \gg R$$

$$\Rightarrow -E_0 R \cos\phi = A_0 + B_0 \ln r + \sum_{l=1}^{\infty} \left(A_l R^l + \frac{B_l}{R^l} \right) (C_l \sin l\phi + D_l \cos l\phi)$$

$$\Rightarrow A_0 = B_0 = C_l = 0 \text{ and } D_l = 0 \text{ for } l \neq 1$$

$$\Rightarrow -E_0 R \cos\phi = \sum_{l=1}^{\infty} \left(A_l R^l + \frac{B_l}{R^l} \right) D_l \cos l\phi$$

Absorbing D_l into A_1 and B_1, we are left with

$$-E_0 R \cos\phi = \left(A_1 R^1 + \frac{B_1}{R^1} \right) \cos\phi \approx A_1 R^1 \cos\phi \Rightarrow A_1 = -E_0$$

Since

$$\varphi = 0 \text{ when } r = R$$

$$\Rightarrow 0 = \left(A_1 R^1 + \frac{B_1}{R^1} \right) \cos\phi \Rightarrow A_1 R = -\frac{B_1}{R}$$

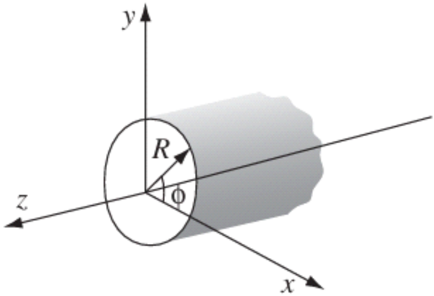
Thus, the potential is

$$\varphi(r, \phi) = \left(A_1 r + \frac{B_1}{r} \right) \cos\phi = \left(-E_0 r + E_0 \frac{R^2}{r} \right) \cos\phi$$

The surface charge is

$$\sigma = -\epsilon_0 \frac{\partial\varphi}{\partial r} = -\epsilon_0 \left(-E_0 - E_0 \frac{R^2}{R^2} \right) \cos\phi = 2\epsilon_0 E_0 \cos\phi$$

(8) A specified charge density $\sigma(\theta) = a \sin 5\phi$ is glued over the surface of an infinite cylinder of radius R. Find the resulting potential inside and outside the cylinder.



ANSWER:

Inside the cylinder:

$$\varphi(r,\phi) = A_0 + \sum_{l=1} r^l \left(C_l \sin l\phi + D_l \cos l\phi\right)$$

Outside the cylinder:

$$\varphi(r,\phi) = A_0' + \sum_{l=1} \frac{1}{r^l} \left(C_l' \sin l\phi + D_l' \cos l\phi\right)$$

Applying the boundary conditions:

$$\frac{\partial \varphi_{\text{out}}}{\partial r} - \frac{\partial \varphi_{\text{in}}}{\partial r} = -\frac{\sigma_0(\theta)}{\epsilon_0}$$

Thus, we have

$$\sum_{l=1} -\frac{l}{R^{l+1}} \left(C_l' \sin l\phi + D_l' \cos l\phi\right)$$

$$-\sum_{l=1} lR^{l-1} \left(C_l \sin l\phi + D_l \cos l\phi\right) = -\frac{a}{\epsilon_0} \sin 5\phi$$

Evidently

$$D_l = D_l' = 0$$

$$C_l = C_l' = 0 \text{ for } l \neq 5$$

$$\Rightarrow \frac{a}{\epsilon_0} = \frac{5}{R^6} C_5' + 5R^4 C_5$$

Since $\varphi_{\text{out}}(R) = \varphi_{\text{in}}(R)$, we have

$$A_0 + R^5 C_5 \sin 5\phi = A_0' + \frac{1}{R^5} C_5' \sin 5\phi$$

$$\Rightarrow A_0 = A_0' \text{ and } R^5 C_5 = C_5'/R^5$$

Here, we can let $A_0 = A_0' = 0$.

Thus, we get

$$a = 5\epsilon_0 \left(\frac{C_5'}{R^6} + R^4 C_5\right) = 10\epsilon_0 R^4 C_5 \Rightarrow C_5 = \frac{a}{10\epsilon_0 R^4}, C_5' = \frac{aR^6}{10\epsilon_0}$$

Therefore, the potential is

$$\varphi(r,\phi)=\begin{cases} r^5\dfrac{a}{10\epsilon_0 R^4}\sin 5\phi=\dfrac{a\sin 5\phi}{10\epsilon_0}\dfrac{r^5}{R^4}, & r<R \\ \dfrac{1}{r^5}\dfrac{aR^6}{10\epsilon_0}\sin 5\phi=\dfrac{a\sin 5\phi}{10\epsilon_0}\dfrac{R^6}{r^5} \quad, & r>R \end{cases}$$